Gepner models and Landau-Ginzburg/sigma-model correspondence.

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1. Introduction

The internal sector of the model is given by certain orbifold of tensor product of $N = 2$ minimal models. This product can be characterized by K dimensional vector

$$\boldsymbol{\mu} = (\mu_1, ..., \mu_K) \tag{1}$$

where

$$\mu_i = 2, 3, ..., \; i = 1, ..., K \tag{2}$$

define the central charges of the individual $N = 2$ minimal models

$$c_i = 3(1 - \frac{2}{\mu_i}) \tag{3}$$

In what follows the μ_i will be specified by

$$\boldsymbol{\mu} = (\mu, \mu, ..., \mu) \tag{4}$$

Then the total central charge is

$$c = \sum_{i=1}^{K} c_i = 3K(1 - \frac{2}{\mu}) \tag{5}$$

There are two cases when the central charge is integer and multiple of 3

$$\mu = K,\ 2K \tag{6}$$

The geometry underlying the first case corresponds to the CY manifold embeded in $\mathbb{P}^{K-1}$ as a degree K hypersurface in $\mathbb{P}^{K-1}$.

In the second case the internal geometry of Gepner model is given by CY manifold which double covers the $\mathbb{P}^{K-1}$ with ramification along certain submanifold.

Generalization:

$$\mu = 3K, 4K, ... \tag{7}$$

The total central charge is no longer integer and these models can not be used as the models of superstring compactification, but the orbifold projection consitent with modular invariance still exists which makes them to be interesting $N=2$ supersymmetic models of CFT from the geometric point of view.

2. Free-field realization of Gepner models.

2.1. Free-field realization of $N=2$ minimal model.

$$X^{*}(z_1)X(z_2) = \ln(z_{12}) + reg.,$$
$$\psi^{*}(z_1)\psi(z_2) = z_{12}^{-1} + reg, \qquad (8)$$

where $z_{12} = z_1 - z_2$. The $N = 2$ super-Virasoro algebra currents are given by

$$\begin{aligned} G^{+}(z) &= \psi^{*}(z)\partial X(z) - \frac{1}{\mu}\partial\psi^{*}(z), \\ G^{-}(z) &= \psi(z)\partial X^{*}(z) - \partial\psi(z), \\ J(z) &= \psi^{*}(z)\psi(z) + \frac{1}{\mu}\partial X^{*}(z) - \partial X(z), \\ T(z) &= \frac{1}{2}(\partial\psi^{*}(z)\psi(z) - \psi^{*}(z)\partial\psi(z)) + \\ &\partial X(z)\partial X^{*}(z) - \frac{1}{2}(\partial^{2}X(z) + \frac{1}{\mu}\partial^{2}X^{*}(z)) \end{aligned} \qquad (9)$$

$N=2$ Virasoro superalgebra is acting naturally in Fock module generated by the fermionic and bosonic operators from the (NS) vacuum state $|p, p^* >$

$$\psi_r|p,p^* >= \psi_r^*|p,p^* >= 0, r \geq \frac{1}{2},$$
$$X_n|p,p^* >= X_n^*|p,p^* >= 0, n \geq 1,$$
$$X_0|p,p^* >= p|p,p^* >,$$
$$X_0^*|p,p^* >= p^*|p,p^* > . \qquad (10)$$

It is a primary state with respect to the generators of $N = 2$ Virasoro algebra

$$G_r^{\pm}|p,p^* >= 0, r \geq \frac{1}{2},$$
$$J_n|p,p^* >= L_n|p,p^* >= 0, n > 0,$$
$$J_0|p,p^* >= \frac{j}{\mu}|p,p^* >= 0,$$
$$L_0|p,p^* >= \frac{h(h+2) - j^2}{4\mu}|p,p^* >= 0, \quad (11)$$

where $j = p^* - \mu p,\ h = p^* + \mu p$.

The Fock modules are reducible representations of $N = 2$ Virasoro superalgebra, while the space of states of minimal models involves only irreducible representations. The irreducible representations are given by cohomology of some complex (known as butterfly complex) building up from Fock modules.

The differential of the complex is given by

$$
\begin{aligned}
Q &= Q^{+} + Q^{-}, \\
Q^{+} &= \oint dz S^{+}(z), \\
S^{+}(z) &= \psi^{*} \exp(X^{*})(z), \\
Q^{-} &= \oint dz S^{-}(z), \\
S^{-}(z) &= \psi \exp(\mu X)(z)
\end{aligned}
\tag{12}
$$

2.2.Free-field realization of the product of minimal models.

$$
X_i(z), X_i^{*}(z),\ \psi_i(z), \psi_i^{*}(z),\ i = 1, ..., K \tag{13}
$$

The $N = 2$ Virasoro superalgebra currents in the product of minimal models are given by

$$
\begin{aligned}
G^{\pm}(z) &= \sum_i G_i^{\pm}(z), \\
J(z) &= \sum_i J_i(z), \\
T(z) &= \sum_i T_i(z),\ i = 1, ..., K
\end{aligned}
\tag{14}
$$

This algebra is acting in the tensor product of Fock modules.

Differential of the complex

$$
\begin{aligned}
S_i^{+}(z) &= \psi_i^* \,\mathrm{exp}(X_i^*)(z), \\
S_i^{-}(z) &= \psi_i \,\mathrm{exp}(\mu X_i)(z), \\
Q_i^{\pm} &= \oint dz S_i^{\pm}(z), \\
Q &= \sum (Q_i^{+} + Q_i^{-})
\end{aligned}
\tag{15}
$$

2.3. The orbifold.

The orbifold group is $\mathbb{Z}_\mu$ and generated by

$$
g = \mathrm{exp}(\imath 2\pi J_0) \tag{16}
$$

3. LG orbifold geometry of Gepner models.

The (internal) space of states of the Gepner model is given by cohomology of certain complex. It is the orbifold of the product of butterfly complexes of minimal models.

The cohomology can be calculated by two steps.

At first step we take the cohomology wrt the operator

$$Q^{+} = \sum_{i=1}^{K} Q_i^{+} \tag{17}$$

It is generated by $bc\beta\gamma$ system of fields

$$\begin{aligned} a_i(z) &= \exp[X_i](z), \\ \alpha_i(z) &= \psi_i \exp[X_i](z), \\ a_i^*(z) &= (\partial X_i^* - \psi_i \psi_i^*) \exp[-X_i](z), \\ \alpha_i^*(z) &= \psi_i^* \exp[-X_i](z) \end{aligned} \tag{18}$$

The N=2 Virasoro superalgebra currents of the model are given by

$$\begin{aligned} G^{-} &= \sum_i \alpha_i a_i^*, \\ G^{+} &= \sum_i (1-\frac{1}{\mu})\alpha_i^* \partial a_i - \frac{1}{\mu} a_i \partial \alpha_i^*, \\ J &= \sum_i (1-\frac{1}{\mu})\alpha_i^* \alpha_i + \frac{1}{\mu} a_i a_i^*, \\ T &= \\ \sum_i \frac{1}{2}((1+\frac{1}{\mu})\partial\alpha_i^*\alpha_i &- (1-\frac{1}{\mu})\alpha_i^*\partial\alpha_i) + \\ (1-\frac{1}{2\mu})\partial a_i a_i^* &- \frac{1}{2\mu} a_i \partial a_i^* \end{aligned} \tag{19}$$

G_0^- is string version of de Rham differential on $\mathbb{C}^K$ if we postulate the following correspondence

$$\begin{aligned} a_i(z) &\leftrightarrow \ coorinates\ a_i, \\ a_i^*(z) &\leftrightarrow \ \frac{\partial}{\partial a_i}, \\ \alpha_i(z) &\leftrightarrow \ da_i, \\ \alpha_i^*(z) &\leftrightarrow \ conjugated\ to\ da_i \end{aligned} \qquad (20)$$

The coordinate changes on $\mathbb{C}^K$ generate isomorphic $bc\beta\gamma$ systems. It endows the $bc\beta\gamma$ system with the structure of sheaf known as chiral de Rham complex due to Malikov and Shechtman.

Making the projection on $\mathbb{Z}_\mu$-invariant states and adding twisted sectors we obtain the chiral de Rham complex of the orbifold

$$\mathbb{C}^K/\mathbb{Z}_\mu,\ \mu = K,\ 2K \qquad (21)$$

The second step is the cohomology with respect to the differential

$$Q^- = \sum_{i=1}^{K} Q_i^- = \oint dz \sum_{i=1}^{K} \alpha_i (a_i)^{\mu-1} \qquad (22)$$

It is equivalent to the restriction of the space of states to the points

$$dW = 0 \tag{23}$$

of the LG potential

$$W = \sum_{i=1}^{K} (a_i)^{\mu} \tag{24}$$

Thus the space of states of the Gepner model is the space of states of LG orbifold $\mathbb{C}^K/\mathbb{Z}_\mu$.

4. LG/sigma-model correspondence.

The orbifold singularity resolution.

4.1. Illustrative example: K=2

$$Q^{+} \to Q^{+} + D_{orb},$$
$$D_{orb} =$$
$$\oint dz \frac{1}{2}(\psi_1^* + \psi_2^*) \exp(\frac{1}{2}(X_1^* + X_2^*))(z) \tag{25}$$

The operator D_{orb} commutes with the Virasoro superalgebra currents and commutes also with the operators Q_i^- when

$$\mu = K, 2K, 3K, ... \tag{26}$$

Then the first step of cohomology calculation splits into 2 substeps.

Substep 1. $Q_1^+ + D_{orb}$*-cohomology.*

It is given by the following $bc\beta\gamma$ fields

$$
\begin{aligned}
b_0(z) &= \exp{[2X_2]}(z),\\
\beta_0(z) &= 2\psi_2 \exp{[2X_2]}(z),\\
b_0^*(z) &= (\frac{1}{2}(\partial X_1^* + \partial X_2^*) - \\
&\psi_2(\psi_1^* + \psi_2^*)) \exp{[-2X_2]}(z),\\
\beta_0^*(z) &= \frac{1}{2}(\psi_1^* + \psi_2^*) \exp{[-2X_2]}(z),\\
b_1(z) &= \exp{[X_1 - X_2]}(z),\\
\beta_1(z) &= (\psi_1 - \psi_2) \exp{[X_1 - X_2]}(z),\\
b_1^*(z) &= \\
&(\partial X_1^* - (\psi_1 - \psi_2)\psi_1^*) \exp{[X_2 - X_1]}(z),\\
\beta_1^*(z) &= \psi_1^* \exp{[X_2 - X_1]}(z)
\end{aligned}
\quad (27)
$$

Substep 2. Q_2^+*-cohomology.*

Another way:
Substep 1. $Q_2^+ + D_{orb}$*-cohomology*
Substep 2. Q_1^+*-cohomology*

Going by this way we obtain another $bc\beta\gamma$ fields:

$$
\begin{aligned}
\tilde{b}_0(z) &= \exp[2X_1](z),\\
\tilde{\beta}_0(z) &= 2\psi_1 \exp[2X_1](z),\\
\tilde{b}^*_0(z) &= (\frac{1}{2}(\partial X^*_1 + \partial X^*_2) -\\
&\psi_1(\psi^*_1 + \psi^*_2)) \exp[-2X_1](z),\\
\tilde{\beta}^*_0(z) &= \frac{1}{2}(\psi^*_1 + \psi^*_2) \exp[-2X_1](z),\\
\tilde{b}_1(z) &= \exp[X_2 - X_1](z),\\
\tilde{\beta}_1(z) &= (\psi_2 - \psi_1) \exp[X_2 - X_1](z),\\
\tilde{b}^*_1(z) &=\\
(\partial X^*_2 - (\psi_2 - \psi_1)\psi^*_2) &\exp[X_1 - X_2](z),\\
\tilde{\beta}^*_1(z) &= \psi^*_2 \exp[X_1 - X_2](z)
\end{aligned}
\tag{28}
$$

$bc\beta\gamma$ fields (27) and (28) correspond to the coordinates of the standard covering of the $O(2)$-bundle total space over $\mathbb{P}^1$.

$$
\begin{gathered}
O(2) = U \cup \tilde{U}\\
U = (b_0, b_1),\ \tilde{U} = (\tilde{b}_0, \tilde{b}_1)
\end{gathered}
\tag{29}
$$

The intersection $U \cap \tilde{U}$

$$
\begin{aligned}
b_0(z) = \tilde{b}_0(z)(\tilde{b}_1)^2(z) &\leftrightarrow b_0 = \tilde{b}_0(\tilde{b}_1)^2,\\
b_1(z) = \tilde{b}^{-1}_1(z) &\leftrightarrow b_1 = \tilde{b}^{-1}_1, \ldots
\end{aligned}
\tag{30}
$$

Therefore in the open set U:

$$
\begin{aligned}
b_0(z) &\leftrightarrow \ coordinate\ along\ the\ fiber, \\
b_1(z) &\leftrightarrow \ coordinate\ along\ the\ base
\end{aligned}
\tag{31}
$$

In the open set $\tilde{U}$:

$$
\begin{aligned}
\tilde{b}_0(z) &\leftrightarrow \ coordinate\ along\ the\ fiber, \\
\tilde{b}_1(z) &\leftrightarrow \ coordinate\ along\ the\ base
\end{aligned}
\tag{32}
$$

The orbifold group acting on the $O(2)$-bundle over $\mathbb{P}^1$ is

$$
\mathbb{Z}_{\mu/2} \subset \mathbb{Z}_{\mu},\ \mu = 2, 4 \tag{33}
$$

It is acting only along the fibers so the base $\mathbb{P}^1$ is the fixed point set of the action.

The potential is given by

$$
W = (b_0)^{\frac{\mu}{2}}(1 + (b_1)^{\mu}) \tag{34}
$$

The $dW = 0$ points (Q^--cohomology) are given by the equations

$$
\begin{aligned}
(b_0)^{\frac{\mu}{2}-1} &= 0,\ when\ b_1^{\mu} \neq -1, \\
(b_0)^{\frac{\mu}{2}} &= 0,\ when\ b_1^{\mu} = -1,
\end{aligned}
\tag{35}
$$

Geometric interpretation.

$\mu = 2$ case.

The equations (35) give the 0-dimensional CY manifold which is given by 2 points in $\mathbb{P}^1$.

$\mu = 4$ case.

The set of solutions of (35) is $\mathbb{P}^1$ with 4 marked points. Hence we have $N = 2$ superconformal σ-model on $\mathbb{P}^1$.

CFT-calculation of the Poincare polynomial:

$$P(t) = 1 + t - t - 1 = 0 \tag{36}$$

it is Poincare polynomial of the torus. Taking into account $\mathbb{Z}_2$-orbifold projection we conclude that $\boldsymbol{\mu} = (4, 4)$ Gepner model corresponds (after the orbifold resolution) to the σ-model on the torus which double cover the $\mathbb{P}^1$ with ramification along the points

$$b_1^4 = -1 \tag{37}$$

4.2. $K > 2$ generalization.

$\mu = K$ case.

When $\mu = K$ we obtain CY manifold embeded in $\mathbb{P}^{K-1}$ as degree K hypersurface (well known fact).

$\mu = 2K$ case.

When $\mu = 2K$ we have $N = 2$ superconformal σ-model on $\mathbb{P}^{K-1}$. It is the CY manifold which double covers $\mathbb{P}^{K-1}$ with ramification along the manifold

$$\sum_{i=1}^{K-1} b_i^{2K} = -1 \tag{38}$$